Using the Water Jugs Puzzle in Management Science Class

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This paper contains a description of an assignment whereby a classic, well-known puzzle is introduced as the subject for spread sheet modeling and solution in a management science class. Results from the classroom experience are discussed.

INTRODUCTION

In this paper, I describe my first-time use of a well-known puzzle in a management science class. The problem is known under various guises as The Water Jugs Puzzle (WJP), the Water Jugs Problem, The Water Jars Problem, or the Die-Hard 3 Problem. Various sources place written versions in the works of Dudeney and Loyd (The die hard jugs, n.d.; Knuth, 2001; Man, 2012) although a search of *Amusements in Mathematics* reveals some similar but not-identical puzzles (c.f. numbers 362, 364, and 365) (Dudeney, 1917). WJPs can be traced at least as far back as the thirteenth century (Bailey, 2008). Similar problems are used in the study of cognitive functioning under the name Water Jug Task (Carder, Handley & Perfect, 2007). The puzzle is considered widely known as is alluded to by its fourth name, above, the Die-Hard 3 Problem. In the 1995 movie *Die Hard with a Vengeance*, characters played by Bruce Willis and Samuel L. Jackson are required to solve a variant of this puzzle in order to prevent a catastrophic event.

Man's (2012) description of the WJP inspired my spreadsheet solution. Further reading on the problem's uses inspired my including it in my curriculum. The problem reads: "You are at the side of a river. You have a 4 liter jug and a 9 liter jug. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure 6 liters of water?" (Man, 2012, p. 109). In Man's other articles (2013a, 2013b), he uses 3 and 5 liter jugs to arrive at 4 liters, as did the screenwriter for *Die Hard with a Vengeance*. Incorporating recreational mathematics into the curriculum should produce an engaging exercise.

A brief literature review follows this section. A section describing the development of my spreadsheet model, along with a detailed description follows. I conclude with a discussion of student progress and lessons learned.

LITERATURE REVIEW

Puzzles and games have been used recreationally for centuries. Henry Ernest Dudeney (1857-1930) was one of the most prolific mathematical puzzle creators of the past two hundred years. *The Canterbury Puzzles and Other Curious Problems* is still in print today (Dudeney, 1907). Eventually, he created five other popular puzzle collections. His range of puzzle-types was exceedingly large, including geometry-based puzzles, logic puzzles, chess problems (Nowlan, n.d.), "combinatorics" (Bremner, 2011), and

"cryptarithmetic" (Kilpelainen, 2012). Many of these puzzles were published in various periodicals under the pseudonym "Sphinx." In addition to Dudeney, some of the better known writers in the field of recreational mathematics include Sam Loyd, Raymond Smullyan, Martin Gardner, and Charles Lutwidge Dodgson, better known as Lewis Carrol, author of *Alice's Adventures in Wonderland*.

Recreational mathematics is the term frequently used to describe mathematical games, puzzles and riddles. The best known peer-reviewed journal devoted to recreational mathematics is the *Journal of Recreational Mathematics* (Baywood Publishing, Inc., n.d.). Many journals give some passing attention to the subject, often through dedicated columns. For example, *Communications of the ACM* regularly publishes a column named "last byte" (Winkler, 2012). Alexander Dewdney (not to be confused with H. E. Dudeney!) wrote a well known column in *Scientific American* during the 1980s, as did Gardner for over twenty-four years prior. Dewdney's column was named "Computer Recreations" while Gardener's column was named "Mathematical Games" (Jimenez & Munoz, 2011, p. 786). The column "Classroom Capsules" appears in *The College Mathematics Journal*. Problems and puzzles often appear in the column to provide an "effective teaching strategy or tool" for college mathematics instruction (Alfaro, Han, & Schilling, 2011, p. 57). In the early twentieth century, Carver (1923) suggested using Dudeney's puzzles as "stimulus" to undertake investigation, for both students and teachers!

"Recreational programming" is described as the practice of studying computer programming by solving problems of a playful nature. The discipline description notes that problems "similar to those of recreational mathematics" are encompassed (Jimenez & Munoz, 2011, p. 775). "Recreational computer science" is the term used by Demain (2010, p. 452). Kino and Uno (2012) briefly discuss the incorporation of computers into the study of games and puzzles, and the reasons therefore. These authors modeled the game Tantrix using an integer programming formulation and solved it with an IBM software product. Kilpelainen (2012) uses problems from recreational mathematics, including some of Dudeney's other puzzles, to assess the features and utility of a new programming language. He notes that the process of formulating puzzle solution algorithms for computation invites programmers to consider problem generalization.

Man (2012), a professor of mathematics, provides a thorough list of types and sources of solutions used in the WJP. He has published three solutions: one based on integer sequencing (2012), one based on a non-heuristic approach (2013a), and one based on arithmetic methods (2013b). I could find no mention of formulating the WJP as a spreadsheet model.

MY SOLUTION

My spreadsheet solution to this problem starts where many optimizing models start: with the notion of what the solution should look like. In this case, the solution should consist of a series or list of moves that result in the correct amount of water in the jugs. An optimum solution will consist of the shortest series of moves, although verifying a global optimum was not a concern at the outset of model formulation.

Given the problem context, there are only six moves! Further, once you consider a particular jug, some moves—say those that concern the other jug—are momentarily not considered, e.g. a move that "undoes" the previous move should not be considered. These facts render the logic imminently programmable. The available moves are defined in Table 1. By the way, this logic is available in the literature (Boldi, Santini, & Vigna, 2002).

Using Table 1, formulating the spreadsheet logic is straightforward. A set of moves is generated and the results are evaluated. In this case, Excel's Solver add-in contains a genetic search algorithm—named Evolutionary—that can generate alternative solutions for evaluation (Frontline Systems, n.d.).

TABLE 1
AVAILABLE MOVES

Move	Action	Result
1	Fill the 4 liter jug from the river.	The 4 liter jug contains 4 liters.
2	Fill the 9 liter jug from the river.	The 9 liter jug contains 9 liters.
3	Pour the contents of the 4 liter jug into the 9 liter jug.	The 9 liter jug contains its original contents plus the contents of the 4 liter jug or the 9 liter jug is filled.
4	Pour the contents of the 9 liter jug into the 4 liter jug.	The 4 liter jug contains its original contents plus the contents of the 9 liter jug or the 4 liter jug is filled.
5	Pour the contents of the 4 liter jug into the river.	The 4 liter jug is empty.
6	Pour the contents of the 9 liter jug into the river.	The 9 liter jug is empty.

My original objective function consisted of minimizing the number of moves required to achieve the desired amount of water, k. During implementation, it was discovered that the genetic algorithm sometimes produced meaningless series of moves, e.g. move 1 followed by move 5. To curtail this event, the objective function was modified to impose a penalty for moves that cancel each other, along with moves that repeat. Thus, the objective function became a goal programming-type of penalty function. Given one row per puzzle move, the formulation is:

> Minimize *Penalty Function* using Solver's Evolutionary setting
> Subject to: (1) moves are integers between 1 and 6 inclusive
> (2) first move is a 1 or a 2
> (3) k liters of water occurs somewhere in the solution.

Since it is assumed to be unknown at the outset how many moves will be required, the move table was arbitrarily set to contain 25 moves. The objective function that was implemented is: *Penalty Function* = (number of rows required to obtain k + number of repeated moves + number of ineffective moves)^3.

The cubic exponent was arbitrarily determined. Of course, hard constraints could be added to force the numbers of repeated and ineffective moves to zero; however, experimentation revealed that this slowed down the algorithm considerably. Also, it often resulted in *Infeasible* results whereas the *Penalty Function* above will usually return a feasible solution of some sort.

Experimentation with the model resulted in modifications to the Evolutionary algorithm default settings in order to avoid *Infeasible* results. Population size was increased to 4500 while the mutation rate was increased to 0.75. Table 2 contains some example performance metrics. Prior to running Solver, the set of moves was randomized using Excel's RANDBETWEEN function. My notebook is a Windows computer with an Intel i5 Core processor while my office PC is a Dell Optiplex 7010 with an i5 Core processor.

TABLE 2
COMPUTER PERFORMANCE METRICS

Run	Note-book time in minutes	Spread-sheet rows required		Office PC time in minutes	Spread-sheet rows required
1	2:44	17		2:32	15
2	3:38	15		4:35	15
3	2:58	14		2:59	**9**
4	2:11	**9**		1:35	10
5	4:17	11		1:37	**9**

Note that the 9 spreadsheet rows required represents an optimal answer. *Infeasible* results do occur, approximately 10 to 15 percent of the time.

STUDENT SOLUTION

Enrollment in the management science course at this School of Business is not large. The course appears in only two degree programs: (1) BS in Logistics and Supply Chain Management as a requirement and (2) the Accounting major of the BBA as an elective, albeit from a limited selection. Twenty students were enrolled. Many of them use e-books. Anecdotally, many of the students are not as spreadsheet literate as I might prefer. As such, the percentage of open notebook and tablet computers in use during class is large. Class meets once per week for three hours in the evening and it is supported by the use of Blackboard.

It is a quiet group; soliciting solution ideas or conversation about solution ideas was not very productive, at least in the early going. At roughly midterm, I began dedicating the first 20 minutes of each class to guided discussion of the problem. I have also required any students who come to my problem-solving sessions to discuss the topic with me although they mostly prefer to discuss graded homework assignments that are due. I have repeated the question "what does the solution look like" numerous times. We have observed the solution to the Travelling Salesman Problem on the Wikipedia page of the same name several times but there are few questions as to how the demonstration applet functions (Travelling Salesman Problem, n.d.). Eventually, I programmed an animation to demonstrate the problem. That applet is located on my youtube channel (Friesen, n.d). One thing that I have not done is provide a solution to the class. The problem was posed in the syllabus with roughly five percent of the available points allocated to formulating a generalizable spreadsheet solution.

A few students "solved" the problem during the first week by providing sets of steps that result in the coveted 6 liters of water. They were unable to provide a generalizable spreadsheet solution at that point in the semester.

In a fruitful development, a student identified a web article associated with the Die-Hard 3 Problem that describes a solution model. The student revealed the contents of (The Die Hard 3 problem, n.d.) which reports that the recipe to solve such a problem may be found by utilizing a theorem that states: *mp + nq = k.* This is a Diophantine equation (Man, 2013b), where *p*=4, *q*=9, and *k*=6. Under certain conditions, solving for the integers *m* and *n* yields information about the solvability of a particular configuration, as well as providing insight into the steps that are required. In particular, if *p* < *q* and if *p* and *q* are ***relatively prime***, then for any integer *k* < *q*, the integers *m* and *n* exist according to the equation. ***Relatively prime*** means that the greatest common divisor is 1; such is the case with 4 and 9 (The Die Hard 3 Problem, n.d.). There are several (*m, n*) pairs for *m4 + n9 = 6* but the pair with the smallest values is (-3, 2). This solution implies that the 4 liter jug will be emptied three times while the 9 liter jug will be filled twice; thus the following solution is shown in Table 3.

TABLE 3
SHORTEST OR OPTIMAL SOLUTION

Move	Water in 4 liter jug	Water in 9 liter jug	Total water
Initial	0	0	0
1 Fill the 9 liter jug	0	9	9
2 Pour the 9 liter jug into the 4 liter jug	4	5	9
3 Dump out the 4 liter jug	0	5	5
4 Pour the 9 liter jug into the 4 liter jug	4	1	5
5 Dump out the 4 liter jug	0	1	1
6 Pour the 9 liter jug into the 4 liter jug	1	0	1
7 Fill the 9 liter jug	1	9	10
8 Pour the 9 liter jug into the 4 liter jug	4	6	10
9 Dump out the 4 liter jug	0	6	6

The total number of moves is nine. What has this information yielded in terms of class progress? That was unclear at midterm, although I am deeply indebted to the student for doing the research and bringing it to class, even if I had to apply a little "instructor-encouragement" to achieve that last action. I did not see any indication that any student's reticence was based on a competitive stance. I have encouraged the class to perform the solution by hand both for *k=6* and for *k=5* (a much easier set of steps). In one class session that occurred after midterm, a student was publicly considering the logic formulation ("should I use *IF* statements?"). The willingness to discuss the problem in spreadsheet terms is progress.

In terms of improving *my* solution, it might be possible to improve my model by seeking for solutions containing the optimal number of steps. Consider that other solutions to this particular Diophantine equation are (6, -2) and (15,-6). Solver can be programmed to generate these solutions; however, there is no way to guarantee that the minimum-step solution will be found. Hypothesizing that a "pour" step must occur between "fill" and "dump" steps, yields a formula for the total number of moves required = 2*(sum of the absolute values of *m* and *n*) – 1. Thus 2*(|-3| + |2|) – 1 = 9 moves, as shown in the solution table. This information could be incorporated into the logic of the model, or into the Solver formulation.

At the end of the semester, I awarded full credit to two students (10% of the class). The first student simply wrote an orderly sequence of pour, fill, and dump activities that eventually arrives at a solution for any reasonable set of jug sizes and desired amount of water. Conditional formatting was used to indicate the answer. The second student implemented something similar although the spreadsheet writes out the steps used. Both students provided generalizable solutions although their search procedure was included implicitly into the spreadsheet logic.

DISCUSSION

In terms of engaging the class, I am not certain that such occurred. This is somewhat ironic in that the puzzle was introduced in an effort to promote engagement. It may be that the level of spreadsheet literacy is too low for a problem of this difficulty. Perhaps some "before" assessment should occur. A thorough literature review on engagement and the effects thereof has not been completed. The term "engagement" appears in our evaluation of teaching, however.

In terms of improving student spreadsheet modeling ability, I am uncertain that such can be attributed to this exercise. After all, we use a number of models in the class. Most of the homework models are implemented in the text; perhaps the level of creativity required makes this problem difficult. At any rate, separating the effects of this task from the rest of the course content is no small order. Topcu (2011) used

a Self-Efficacy for Math assessment in an analysis of algebra teaching methods; perhaps a similar assessment exists for spreadsheets.

CONCLUSION

There is a large body of problems from recreational mathematics and recreational computing that may be used in quantitative methods classes such as my management science class. The effect upon student engagement is likely related to the instructor's ability to implement the problem as a teaching technique. More study in this area is required.

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